CONDITIONAL STABILITY IN THE CRITICAL CASE OF SEVERAL PAIRS OF PURELY IMAGINARY ROOTS

PMM Vol. 31, No. 3, 1967, pp. 453-459

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(Received November 23, 1966)

We consider (Sections 2 and 3) a problem of constructing regions of conditional stability ([1], par. 1) in the critical case given in the title, using two Liapunov – Chetaev functions [2 and 3]. A new form of equations of perturbed motion first obtained by Kamenkov in [4] is used here. In Section 4 a method of constructing regions of conditional stability for systems with time delay, is presented. Finally, in Section 5, an example is given

1. Unperturbed motion is conditionally stable if it is stable under initial perturbations constrained by conditions of the type

$$f(x_1, \ldots, x_n) = 0 \quad \text{or} \quad f(x_1, \ldots, x_n) > 0$$

where f is a function of perturbations $x_1, \ldots, x_n$ and $f(0, \ldots, 0) = 0$ (see [1], par. 1).

A problem of particular interest is that of defining the regions of conditional stability in critical cases when either the unperturbed motion is unstable or when available criteria are insufficient to solve the problem of stability. In these cases a problem arises of stabilizing the unperturbed motion by suitable selection of initial perturbations. It was Liapunov who first demonstrated the existence ([1], par. 24) and method of constructing regions of conditional stability, when investigating a critical case of a double zero root, for a group of solutions [5 and 7]. Problems of existence of such regions were investigated in [7 to 10]. Possibility of successful construction of such regions of conditional stability in critical cases follows, in our opinion, from the basic Chetaev theorem [2], where for the first time a method of solving stability problems using several Liapunov functions [3] is shown. In a number of critical cases, two Liapunov-Chetaev functions were successfully used to construct regions of conditional stability [11 and 12]. Below we use the same method to investigate the conditional stability in the critical case of k pairs of simple, purely imaginary roots, and a novel form of equations proposed in [4] allows us to obtain these regions in a fairly general form.

2. Let us consider a system of equations of perturbed motion in the case when the characteristic equation of the linear first approximation system has a $(n+1)$-th pair of simple purely imaginary roots $\pm i\lambda_s$, $\lambda_s = \text{const} > 0$ ($s = 0, 1, \ldots, n$) and l roots with negative real parts. We shall assume that the coefficients of nonlinear right-hand side terms are periodic functions of t and are all of the same period 2π. Then, a possible form which the system can take (on assumption that variables are chosen in such a way that the critical part of the system has a canonical form), is

$$\frac{d\mathbf{x}}{dt} = Q\mathbf{x} + X(\mathbf{x}, \mathbf{y}, t), \qquad \frac{d\mathbf{y}}{dt} = P\mathbf{y} + Y(\mathbf{x}, \mathbf{y}, t) \tag{2.1}$$

A constant $2(m+1) \times 2(n+1)$ matrix Q has the form

$$Q = \{Q_0, Q_1, \ldots, Q_n\}, \qquad Q_s = \begin{bmatrix} 0 & -\lambda_s \\ \lambda_s & 0 \end{bmatrix}$$

Here, x is a $2(n+1)$-dimensional vector and $\mathbf{y}$ is an l-vector; matrix P is of the order $l \times l$, is stable and constant; $X(\mathbf{x}, \mathbf{y}, t)$ and $Y(\mathbf{x}, \mathbf{y}, t)$ are vector functions of respective dimensions and their components are power series in coordinates of $\mathbf{x}$- and $\mathbf{y}$-vectors with coefficients periodic in t and of period 2π. These series begin with terms of order not higher than second, and they converge in the region

$$|\mathbf{x}| \leqslant H, \quad |\mathbf{y}| \leqslant H, \quad t \geqslant t_0 \quad (H = \text{const} > 0, \ |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_{2n+1}^2})$$

Let us assume that solution of the problem of constructing regions of conditional stability for an abbreviated system [11], is independent of terms of order higher than N.

We assume that irrational numbers λ_s are not connected by a relation $m_0\lambda_0 + \ldots + m_n\lambda_n = 0$ when $|m_0| + \ldots + |m_n| \leqslant N$ where m_s are integers. Using the transformations of ([13], Section 97) we can reduce (2.1) to a similar form, but the expansion of vector function $Y(\mathbf{x}, 0, t)$ will then begin with a term of order not lower than $N + 1$, and terms of the expansion of $X(\mathbf{x}, 0, t)$ of up to and including the N-th order will not contain t explicitly. Applying further the reduction principle to the case of conditional stability [11] we can reduce the problem of constructing regions of conditional stability to an analogous problem for the abbreviated part of system (2.1)

$$\frac{d\mathbf{x}}{dt} = Q\mathbf{x} + X^{(2)}(\mathbf{x}) + \cdots + X^{(N)}(\mathbf{x}) + \varphi(\mathbf{x}, t), \quad |\varphi(\mathbf{x}, t)| < A|\mathbf{x}|^{N+1} \tag{2.2}$$

Here components of the vector $X^{(r)}(\mathbf{x})$ are forms which are of r-th order in coordinates of the vector $\mathbf{x}$ and A is a positive integer. A set of transformations given in [4] can be used now to reduce the problem of stability of (2.2) in the case when solution of the stability problem can be given in a finite number of terms, to investigation of stability of unperturbed motion of the system

$$\frac{dr_s}{dt} = r_s \sum_{i=0}^{n} a_{si} r_i^2 + r_s \sum a_s r_0^{2k_0} \cdots r_n^{2k_n} + R_s(r_0, \ldots, r_n, \vartheta_0, \ldots, \vartheta_n, t)$$

$$\left(4 \leqslant \sum_{i=0}^{n} 2k_i \leqslant N, \ s = 0, 1, \ldots, n\right) \tag{2.3}$$

where a_{si} and a_s are real constants and expansions R_s in r_i begin with terms of order not lower than $(N + 1)$. Later, regions of conditional stability will be constructed for (2.3). Were these regions constructed for the system (2.2), then the problem of existence of analogous regions for the initial system would reduce to the problem of finding a solution for a system of nonlinear inequalities and that, as we know, would be difficult. It is for that reason, that the initial system should be brought to the form (2.3) when regions of conditional stability are constructed in the critical case under consideration.

3. Without loss of generality, we can construct regions of conditional stability near the r_n-axis. We can take the Liapunov-Chetaev function for (2.3) in the form

$$2V = r_0^2 + r_1^2 + \ldots + r_n^2, \qquad 2W(k) = r_0^2 - k(r_1^2 + \ldots + r_n^2) \tag{3.1}$$

where the constant $k \geqslant 0$ will be defined later.

Derivatives of (3.1) in t, by (2.3) are of the form

$$\frac{dV}{dt} = r_0^2 \sum_{s=0}^{n} a_{0s} r_s^2 + r_0^2 \sum_{i=1}^{n} a_{i0} r_i^2 + \sum_{i,j=1}^{n} a_{ij} r_i^2 r_j^2 + R^{(1)}(r_0^2, \ldots, r_n^2, t)$$

$$\frac{dW}{dt} = r_0^2 \sum_{s=0}^{n} a_{0s} r_s^2 - k r_0^2 \sum_{i=1}^{n} a_{i0} r_i^2 - k \sum_{i,j=1}^{n} a_{ij} r_i^2 r_j^2 + R^{(2)}(r_0^2, \ldots, r_n^2, t)$$

Here $R^{(1)}$ and $R^{(2)}$ begin with terms of order not lower than the sixth in r_s. Then

$$\left(\frac{dV}{dt}\right)_{W(k)=0} = a_{00}k^2\left(\sum_{i=1}^{n} r_i^2\right)^2 + k\sum_{i=1}^{n} r_i^2 \sum_{j=1}^{n}(a_{0j}+a_{j0})\,r_j^2 + $$
$$+\sum_{i,\,j=1}^{n} a_{ij}r_i^2r_j^2 + R^{(3)}(r_1^2,\ldots,r_n^2,t) \tag{3.2}$$

Function $R^{(3)}$ resembles $R^{(1)}$ and $R^{(2)}$. Suppose that (3.2) assumes nonpositive values when $k \in [0, k_1]$ and r_0^2 are sufficiently small. Sufficient conditions for this can be obtained as follows.

We introduce a n-dimensional vector $\mathbf{r} = \{r_1^2,\ldots, r_n^2\}$ and $B(k) = \| b_{ij}(k)\|^n_1$, the latter being a matrix whose elements are

$$b_{ij}(k) = {}^1/_2\,[2a_{00}k^2 + k(a_{0i}+a_{i0}+a_{0j}+a_{j0}) + (a_{ij}+a_{ji})]$$

Principal minors of the matrix $B(k)$ will be denoted by

$$B_1(k),\ B_2(k),\ldots,\ B_n(k)$$

Let us now denote by k_1, a maximum value of k such, that when $k \in [0, k_1]$.

$$(-1)^i B_i(k) > 0 \qquad (i = 1, 2, \ldots, n) \tag{3.3}$$

Obviously, the requirement that the form $\sum a_{ij}r_i^2r_j^2$ be negative definite is sufficient for conditions (3.3) to hold. Consequently, by continuity there exists an interval $[0, k_1]$ such that when $k \in [0, k_1]$, then the form $\mathbf{r}'B(k)\,\mathbf{r}$ is negative definite. Suppose now, that con ditions (3.3) are fulfilled. Then, a sufficiently small h_1 = const > 0 can be found such that

$$\operatorname{sign}\left(\frac{dV}{dt}\right)_{W(k)=0} = \operatorname{sign}\,[\mathbf{r}'B(k)\,\mathbf{r}] \tag{3.4}$$

when $k \in [0, k_1]$ and $r_0^2 \leqslant h_1$. Function (3.2) will be negative definite. When $W(k_1) = 0$,

$$\left(\frac{dW}{dt}\right)_{W(k_1)=0} = \mathbf{r}'C(k_1)\,\mathbf{r} + R^{(4)}(r_1^2,\ldots,r_n^2,t) \tag{3.5}$$

Function $R^{(4)}$ is similar in form to $R^{(3)}$, and

$$C(k_1) = \| c_{ij}(k_1)\|_1^n$$

$$c_{ij}(k_1) = {}^1/_2\,k_1\,[k_1(2a_{00} - a_{i0} - a_{j0}) + (a_{0i} + a_{0j} - a_{ij} - a_{ji})]$$

Let us now stipulate that (3.5) assumes positive values. A sufficient condition for this is, that principal minors $C_1(k_1)$, $C_2(k_1)$, ..., $C_n(k_1)$ of the matrix $C(k_1)$ satisfy the inequalities

$$C_i(k_1) > 0 \qquad (i = 1, 2, \ldots, n) \tag{3.6}$$

and $r_0^2 \leqslant h_2$ where h_2 = const > 0 is sufficiently small. The functions (3.2) and (3.5) are sign definite, provided that (3.3), (3.6) and $r_0^2 \leqslant h = \min(h_1, h_2)$ all hold. Function (3.2) is negative definite, while (3.5) is positive definite. By [3], unperturbed motion of the system (2.3) and, consequently, [11] of (2.1), is asymptotically stable when initial perturbations satisfy the conditions

$$r_0^2(t_0) - k_1\,[r_1^2(t_0) + \ldots + r_n^2(t_0)] > 0, \qquad r_0^2(t_0) \leqslant h \tag{3.7}$$

If function (3.2) assumes nonpositive values and (3.5) is positive when $r_0^2 \leqslant h$, then the nonperturbed motion of the system (2.3) is stable under the initial perturbations (3.7).

Thus, construction of regions of conditional stability (e.g. near the r_0-axis), reduces to the following.

1). We find the maximum value of k_1 such, that when $k \in [0, k_1]$ and $r_0^2 \leqslant h$, then the function (3.2) is negative definite (e.g. when conditions (3.3) hold).

2). We stipulate that when $k = k_1$ and $r_0^2 \leqslant h$, the the function (3.5) is positive (e.g. conditions (3.6)). Then the unperturbed motion of the system (2.3) will be asymptotically stable under the initial perturbations (3.7).

If the function (3.2) is always positive, then the unperturbed motion of (2.3) will be stable under the initial perturbations (3.7).

Note 3.1. If the function (3.5) is positive when $0 < k_2 < k_1$, then the region $W(k_2) > 0$ should be chosen as the region of conditional stability. Obviously, the higher the value of k at which the above conditions hold, the wider the region of stability.

Note 3.2. When $a_{00} > 0$ and the form $\sum a_{ij} r_i{}^2 r_j{}^2$ is negative definite, then the following estimate is valid for k_1

$$0 < k_1 < \frac{1}{2a_{00}} [\min_i (a_{0i} + a_{i0}) + \sqrt{(\min_i |a_{i0} + a_{i0}|)^2 - 4a_{00}\lambda_{\max}}] \tag{3.8}$$

provided that the expression within square brackets is positive (here $\lambda_{\max}$ is the largest eigen number of the matrix $\| a_{ij} \|^n{}_1$).

Indeed, solving

$$a_{00}k^2 \left(\sum_{i=1}^{n} r_i{}^2 \right)^2 + k \sum_{i=1}^{n} r_i{}^2 \sum_{j=1}^{n} (a_{0j} + a_{j0}) r_j{}^2 + \sum_{i,\, j=1}^{n} a_{ij} r_i{}^2 r_j{}^2 = 0$$

for k and applying the estimates ([14], Chapt. X, Section 7), we obtain (3.8).

4. Let a perturbed motion of a system be described by the following Eq. with time delay

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) + A_\tau \mathbf{x}(t - \tau) + X[\mathbf{x}(t), \mathbf{x}(t - \tau)] \tag{4.1}$$

Meaning of the symbols used was thoroughly discussed in [15], hence we shall not repeat it here. It was shown in ([16], Section 29) that 3q. (4.1) has, in the space $C_{[-\tau,\, 0]}$, a corresponding differential operator Eq.

$$\frac{dx_t(\vartheta)}{dt} = P\mathbf{x}_t(\vartheta) + R[\mathbf{x}_t(0), \mathbf{x}_t(-\tau)] \tag{4.2}$$

Let the Eq.

$$\det [A - \lambda E + A_\tau e^{-\lambda\tau}] = 0 \tag{4.3}$$

defining the spectrum of the operator P, have an $(n + 1)$-th pair of purely imaginary roots and remaining roots with negative real parts. Then, using the transformations of [15], we can reduce (4.2) to

$$d\mathbf{v} / dt = Q\mathbf{v} + F[\mathbf{v}, \mathbf{z}_t(0), \mathbf{z}_t(-\tau)]$$

$$\frac{d\mathbf{z}_t(\vartheta)}{dt} = P\mathbf{z}_t(\vartheta) + Z[\mathbf{v}, \mathbf{z}_t(0), \mathbf{z}_t(-\tau), \vartheta] \tag{4.4}$$

where the operator $Z[\mathbf{v}, \mathbf{z}_t(0), \mathbf{z}_t(-\tau), \vartheta]$ satisfies the inequality

$$|Z(\mathbf{v}, 0, 0, \vartheta)| < L|\mathbf{v}|^{N+1} \tag{4.5}$$

where $L = \text{const} > 0$ and the integer N is defined below.

Let us now assume that unperturbed motion of the system

$$\frac{d\mathbf{v}}{dt} = Q\mathbf{v} + F(\mathbf{v}, 0, 0) \tag{4.6}$$

is stable or asymptotically st able under initial perturbations $\mathbf{v}(t_0) \in G$ (closure of the region G contains a stagnation point) and, that it is independent of the terms of order higher than N (in the sense of [11]). Now, if the operator Z satisfies (4.5), then the unperturbed motion of (4.4) is, respectively, stable or asymptotically stable under initial perturbations $\mathbf{v}(t_0) \in G$. Proof of this is analogous to that in [15]. From this it follows that when regions of conditional stability are constructed for the critical cases of systems with time delay, results of [5, 11 and 12] can be utilised together with Section 3 of the present work.

Note 4.1. Let Eq. (4.3) have one zero root and let the remaining roots have negative real parts. Then (4.6) assumes the form

$$\frac{dv}{dt} = gv^m + g_1 v^{m+1} + \dots$$

where v is a scalar, while g, g_1, ... are constants. Let $g \neq 0$ and let the operator Z satisfy the condition (4.5) ($N = m$).

When m is even, unperturbed motion is unstable [17] although at the same time it is conditionally stable, namely asymptotically stable for initial perturbations $gv(t_0) < 0$.

5. Example 1. Let us consider conditional stability of the principal axis of a gyro-horizon rotor with respect to the local vertical. Center of gravity of the gyro-horizon is displaced relative to the axis of suspension. Since up to the moment of switching on the device was arrested, we must assume that angles of deviation Ψ and ϑ of the principal axis of the gyroscope from the local vertical were small at the moment of release. Equations of motion of the principal axis of the gyro-horizon rotor have the form [18 and 19]

$$I_B\vartheta^{\cdot\cdot} + I\Omega(\Psi^{\cdot\cdot} + \omega_c) = -Gl\vartheta + M_+^{(1)} - M_-^{(1)} \operatorname{sign} \vartheta^{\cdot}$$

$$I_c\Psi^{\cdot\cdot} - I\Omega(\vartheta^{\cdot} + \omega_B) = Gl\Psi + M_+^{(2)} - M_-^{(2)} \operatorname{sign} \Psi^{\cdot}$$

$$I_1\varepsilon_1^{\cdot\cdot} + k_1\varepsilon_1 = -M^{(3)} \operatorname{sign} \varepsilon_1^{\cdot}$$

$$I_2\varepsilon_2^{\cdot\cdot} + k_2\varepsilon_2 = -M^{(4)} \operatorname{sign} \varepsilon_2^{\cdot} \qquad (5.1)$$

Since frictional moments are small compared with the moments $M_+^{(1)}$ and $M_+^{(2)}$ of compensating motors and have no influence on the character of the motion of the principal axis, they shall be neglected in the following. Rotation of Earth leads to appearance of small constant terms in (5.1) and they can be eliminated by the following substitution

$$\Psi_1 = \Psi + a, \ \vartheta_1 = \vartheta + e \ \ (a, \ e = \text{const}).$$

Errors caused by the above assumptions are directly proportional to friction. Neglecting nutation terms and approximating $M_+^{(1)}$ and $M_+^{(2)}$ by

$$M_+^{(1)} = -[q_1(\Psi - \varepsilon_1)^3 + q_2(\Psi - \varepsilon_1)^5 + \dots], \quad M_+^{(2)} = h_1(\vartheta - \varepsilon_2)^3 + h_2(\vartheta - \varepsilon_2)^5 + \dots$$

(where q_1, q_2, ..., h_1 and h_2 are constants). After a number of transformations we obtain a set of Eqs.

$$\frac{dr_1}{dt} = \frac{3}{4} r_1 \left[\frac{1}{2}(a_1 + a_2) r_1^2 + a_1 r_2^2 + a_2 r_3^2\right] + \dots \quad \frac{dr_2}{dt} = 0, \quad \frac{dr_3}{dt} = 0 \qquad (5.2)$$

where

$$a_1 = -\frac{q_1}{I\Omega}, \qquad a_2 = -\frac{h_1}{I\Omega}$$

Since the right-hand part of the first Eq. of (5.2) becomes identically zero when $r_1 = 0$, then r_1 either retains its initial sign or $r_1(t) \equiv 0$ when $t \geqslant T \geqslant t_0$ and $r_1(T) = 0$.

Case $r_1(t) \equiv 0$ is uninteresting from the point of view of conditional stability (unperturbed motion is stable when $t \geqslant T$), hence we shall only consider the case when $r_1(t) > 0$ with $r_1(t_0) > 0$ and $t \geqslant t_0$.

We shall seek a region of conditional stability adjacent to the r_3-axis

$$2W(k) = r_3^2 - k(r_1^2 + r_2^2)$$

Expressions (3.2) and (3.5) in this case, become,

$$\left(\frac{dV}{dt}\right)_{W(k)=0} = \frac{3}{4} r_1^2 \left\{\frac{1}{2}[a_1 + (1 + 2k) a_2] r_1^2 + (a_1 + ka_2) r_2^2\right\} + \dots \qquad (5.3)$$

$$\left(\frac{dW}{dt}\right)_{W(k)=0} = -\frac{3k}{4} r_1^2 \left\{\frac{1}{2}[a_1 + (1 + 2k) a_2] r_1^2 + (a_1 + ka_2) r_2^2\right\} + \dots \qquad (5.4)$$

Let

$$a_1 < -a_2 < 0 \qquad (5.5)$$

and let us choose

$$0 < k_1 < -\frac{1}{2}\left[\frac{a_1}{a_2} + 1\right] \qquad (5.6)$$

Terms of (5.3) and (5.4) containing expressions of higher order are such, that the signs of (5.3) and (5.4) are defined by these terms when $r_3{}^2 \leqslant h$, $r_1(t_0) > 0$ and $h = \text{const} > 0$. Then, with (5.5), (5.6) and $k \in [0, k_1]$, the inequalities

$$\left(\frac{dV}{dt}\right)_{W(k)\geqslant 0} < 0, \quad \left(\frac{dW}{dt}\right)_{W(k_1)=0} > 0$$

hold. Consequently, unperturbed motion of the system (5.2) is stable under initial perturbations

$$r_3^2(t_0) - k_1 [r_1^2(t_0) + r_2^2(t_0)] > 0, \qquad r_3^2(t_0) \leqslant h$$

provided that conditions (5.5) and (5.6) hold.

We should note that conditional stability is a result of intersecting constraints.

Example 2. Region of stability coinciding with that shown by Veretennikov, can be obtained as follows.(*) Consider the functions

$$2V = \sum_{i=1}^{3} r_i^2, \qquad 2W(k) = r_3^2 - kr_2^2$$

By (5.2) we have

$$\left(\frac{dV}{dt}\right)_{W(k)=0} = \frac{3}{4} r_1^2 \left[\frac{1}{2}(a_1 + a_2) r_1^2 + (a_1 + ka_2) r_2^2\right] + \ldots, \quad \frac{dW}{dt} \equiv 0$$

Put $k = 1$. When $W(t_0) = r_3{}^2(t_0) - r_2{}^2(t_0) > 0$, the trajectory of solution of (5.2) cannot intersect the surface $W = r_3{}^2 - r_2{}^2 = 0$, since along this trajectory we have $W' \equiv 0$. Consequently, when $a_1 < -a_2 < 0$, unperturbed motion of the system (5.2) is stable under initial perturbations $r_3(t_0) > r_2(t_0)$.

*) **Note.** V.G. Veretennikov's dissertation "Stability of motion in presence of three purely imaginary roots", Moscow, Lumumba University of Friendship of Nations, 1966

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Translated by L. K.